MARKSCHEME

November 2004

MATHEMATICS

Higher Level

Paper 2

27 pages

Paper 2 Markscheme

Note: Where there are 2 marks (*e.g.* M2, A2) for an answer do NOT split the marks unless otherwise instructed

1 Method of marking

(a) All marking must be done using a **red** pen.

(b) Marks should be noted on candidates' scripts as in the markscheme:

- show the breakdown of individual marks using the abbreviations ***(M1)***, ***(A2)*** *etc*., unless a part is completely correct;
- write down each part mark total, indicated on the markscheme (for example, ***[3 marks]***) – it is suggested that this be written at the end of each part, and underlined;
- write down and circle the total for each question at the end of the question.

2 Abbreviations

The markscheme may make use of the following abbreviations:

(M) Marks awarded for **Method**

(A) Marks awarded for an **Answer** or for **Accuracy**

(N) Marks awarded for correct answers, if **no** working (or no relevant working) shown: they may not be all the marks for the question. Examiners should only award these marks for correct answers where there is no working.

(R) Marks awarded for clear **Reasoning**

(AG) **Answer Given** in the question and consequently marks are **not** awarded

Note: Unless otherwise stated, it is not possible to award ***(M0)(A1)***.

Examiners should use **(d)** to indicate where discretion has been used. It should only be used for decisions on follow through and alternative methods. It must be accompanied by a brief note to explain the decision made

Follow through (ft) marks should be awarded where a correct method has been attempted but error(s) are made in subsequent working which is essentially correct.

- Penalize the error when it first occurs
- Accept the incorrect result as the appropriate quantity in all subsequent working
- If the question becomes much simpler then use discretion to award fewer marks
- Use **(d)** to indicate where discretion has been used. It should only be used for decisions on follow through and alternative methods. It must be accompanied by a brief note to explain the decision made.

3 Using the Markscheme

(a) This markscheme presents a particular way in which each question may be worked and how it should be marked. **Alternative methods** have not always been included. Thus, if an answer is wrong then the working must be carefully analysed in order that marks are awarded for a different method in a manner which is consistent with the markscheme. Indicate the awarding of these marks by **(d)**.

Where alternative methods for complete questions or parts of questions are included, they are indicated by **METHOD 1**, **METHOD 2**, *etc*. Other alternative part solutions are indicated by **EITHER….OR**. It should be noted that ***G*** marks have been removed, and GDC solutions will not be indicated using the **OR** notation as on previous markschemes.

Candidates are expected to show working on this paper, and examiners should not award full marks for just the correct answer. Where it is appropriate to award marks for correct answers with no working (or no relevant working), it will be indicated on the markscheme using the ***N*** notation. All examiners will be expected to award marks accordingly in these situations.

(b) Unless the question specifies otherwise, accept **equivalent forms**. For example: $\frac{\sin\theta}{\cos\theta}$ for $\tan\theta$. On the markscheme, these equivalent numerical or algebraic forms will generally be written in brackets after the required answer. Paper setters will indicate the required answer, by allocating full marks at that point. Further working should be ignored, even if it is incorrect. For example: if candidates are asked to factorize a quadratic expression, and they do so correctly, they are awarded full marks. If they then continue and find the roots of the corresponding equation, do not penalize, even if those roots are incorrect, *i.e.* once the correct answer is seen, ignore further working.

(c) As this is an international examination, all **alternative forms of notation** should be accepted. For example: 1.7, $1\cdot 7$, 1,7; different forms of vector notation such as $\vec{u}$, $\bar{u}$, $\underline{u}$; $\tan^{-1} x$ for arctan x.

4 Accuracy of Answers

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy.

There are two types of accuracy error. Candidates should be penalized **once only IN THE PAPER** for an accuracy error **(AP)** - either a rounding error, or a level of accuracy error.

Award the marks as usual then write –1**(AP)** against the answer and also on the **front** cover

Rounding errors: only applies to final answers not to intermediate steps.

Level of accuracy: when this is not specified in the question the general rule *unless otherwise stated in the question all numerical answers must be given exactly or to three significant figures* applies.

- If a final correct answer is incorrectly rounded, apply the **AP**
  OR
- If the level of accuracy is not specified in the question, apply the **AP** for final answers not given to 3 significant figures. (Please note that this has changed from 2003).

Note: **If there is no working shown**, and answers are given to the correct two significant figures, apply the **AP**. However, do **not** accept answers to one significant figure without working.

5 Graphic Display Calculators

Many candidates will be obtaining solutions directly from their calculators, often without showing any working. They have been advised that they must use mathematical notation, not calculator commands when explaining what they are doing. Incorrect answers without working will receive no marks. However, if there is written evidence of using a graphic display calculator correctly, method marks may be awarded. Where possible, examples will be provided to guide examiners in awarding these method marks.

Examples

1 Accuracy

A question leads to the answer 4.6789….

- 4.68 is the correct 3 s.f. answer.
- 4.7, 4.679 are to the wrong level of accuracy: both should be penalized the first time this type of error occurs.
- 4.67 is incorrectly rounded – penalize on the first occurrence.

Note: All these "incorrect" answers may be assumed to come from 4.6789…, even if that value is not seen, but previous correct working is shown. However, 4.60 is wrong, as is 4.5, 4.8, and these should be penalized as being incorrect answers, not as examples of accuracy errors.

2 Alternative solutions

The points P, Q, R are three markers on level ground, joined by straight paths PQ, QR, PR as shown in the diagram. $\text{QR} = 9\text{ km}$, $\text{P}\hat{\text{Q}}\text{R} = 35^\circ$, $\text{P}\hat{\text{R}}\text{Q} = 25^\circ$.
(Note: in the original question, the first part was to find PR = 5.96)

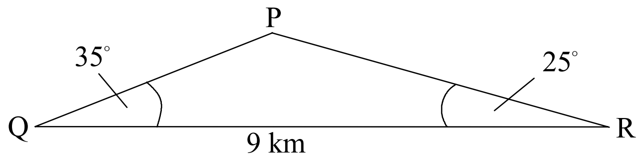
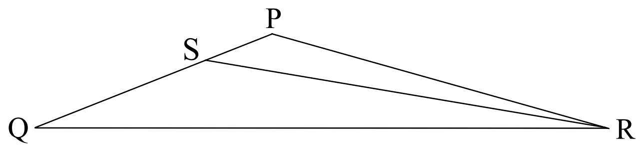


diagram not to scale

(a) Tom sets out to walk from Q to P at a steady speed of 8 km h^{-1}. At the same time, Alan sets out to jog from R to P at a steady speed of $a\text{ km h}^{-1}$. They reach P at the same time. Calculate the value of *a*. ***[7 marks]***

(b) The point S is on [PQ], such that $\text{RS} = 2\text{QS}$, as shown in the diagram.



Find the length QS. ***[6 marks]***

MARKSCHEME

(a) **EITHER**

Sine rule to find PQ

$\text{PQ} = \dfrac{9\sin 25}{\sin 120}$ ***(M1)(A1)***

$\text{PQ} = 4.39 \text{ km}$ ***(A1)***

OR

Cosine rule: $\text{PQ}^2 = 5.96^2 + 9^2 - (2)(5.96)(9)\cos 25$ ***(M1)(A1)***

$= 19.29$

$\text{PQ} = 4.39 \text{ km}$ ***(A1)***

THEN

Time for Tom $= \dfrac{4.39}{8}$ ***(A1)***

Time for Alan $= \dfrac{5.96}{a}$ ***(A1)***

Then $\dfrac{4.39}{8} = \dfrac{5.96}{a}$ ***(M1)***

$a = 10.9$ ***(A1)*** ***(N5)***

[7 marks]

Note that the **THEN** part follows both **EITHER** and **OR** solutions, and this is shown by the alignment.

(b) **METHOD 1**

$\text{RS}^2 = 4\text{QS}^2$ ***(A1)***

$4\text{QS}^2 = \text{QS}^2 + 81 - 18 \times \text{QS} \times \cos 35$ ***(M1)(A1)***

$3\text{QS}^2 + 14.74\text{QS} - 81 = 0$ (or $3x^2 + 14.74x - 81 = 0$) ***(A1)***

$\Rightarrow \text{QS} = -8.20$ or $\text{QS} = 3.29$ ***(A1)***

therefore QS = 3.29 ***(A1)***

METHOD 2

$\dfrac{\text{QS}}{\sin \text{S}\hat{\text{R}}\text{Q}} = \dfrac{2\text{QS}}{\sin 35}$ ***(M1)***

$\Rightarrow \sin \text{S}\hat{\text{R}}\text{Q} = \dfrac{1}{2}\sin 35$ ***(A1)***

$\text{S}\hat{\text{R}}\text{Q} = 16.7^\circ$ ***(A1)***

Therefore $\text{Q}\hat{\text{S}}\text{R} = 180 - (35 + 16.7) = 128.3^\circ$ ***(A1)***

$\dfrac{9}{\sin 128.3} = \dfrac{\text{QS}}{\sin 16.7} \left(= \dfrac{\text{SR}}{\sin 35} \right)$ ***(M1)***

$\text{QS} = \dfrac{9\sin 16.7}{\sin 128.3} \left(= \dfrac{9\sin 35}{2\sin 128.3} \right) = 3.29$ ***(A1)*** ***(N2)***

If candidates have shown no working award ***(N5)*** for the correct answer 10.9 in part (a) and ***(N2)*** for the correct answer 3.29 in part (b).

[6 marks]

3 Follow through

Question

Calculate the acute angle between the lines with equations

$$\boldsymbol{r} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} + s\begin{pmatrix} 4 \\ 3 \end{pmatrix} \text{ and } \boldsymbol{r} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Markscheme

Angle between lines = angle between direction vectors. (May be implied.) ***(A1)***

Direction vectors are $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. (May be implied.) ***(A1)***

$\begin{pmatrix} 4 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \left|\begin{pmatrix} 4 \\ 3 \end{pmatrix}\right| \left|\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right| \cos\theta$ ***(M1)***

$4 \times 1 + 3 \times (-1) = \sqrt{(4^2 + 3^2)}\sqrt{(1^2 + (-1)^2)}\cos\theta$ ***(A1)***

$\cos\theta = \dfrac{1}{5\sqrt{2}}$ $(= 0.1414\ldots)$ ***(A1)***

$\theta = 81.9^\circ$ (1.43 radians) ***(A1)*** ***(N3)***

Examples of solutions and marking

Solutions	Marks allocated
1. $\begin{pmatrix} 4 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \left|\begin{pmatrix} 4 \\ 3 \end{pmatrix}\right| \left|\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right| \cos\theta$	***(A1)(A1)*** implied ***(M1)***
$\cos\theta = \dfrac{7}{5\sqrt{2}}$	***(A0)(A1)***
$\theta = 8.13^\circ$	***(A1)*****ft**

Total 5 marks

Solutions	Marks allocated
2. $\cos\theta = \dfrac{\begin{pmatrix} 4 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 4 \end{pmatrix}}{\sqrt{17}\sqrt{20}}$	***(A0)(A0)*** wrong vectors implied ***(M1)*** for correct method, ***(A1)*****ft**
$= 0.2169$	***(A1)*****ft**
$\theta = 77.5^\circ$	***(A1)*****ft**

Total 4 marks

Solutions	Marks allocated
3. $\theta = 81.9^\circ$	***(N3)***

Total 3 marks

Note that this candidate has obtained the correct answer, but not shown any working. The way the markscheme is written means that the first 2 marks may be implied by subsequent correct working, but the other marks are only awarded if the relevant working is seen. Thus award the first 2 implied marks, plus the final mark for the correct answer.

END OF EXAMPLES

1. (a) $z^n = \cos n\theta + \mathrm{i}\sin n\theta$

$\frac{1}{z^n} = \cos(-n\theta) + \mathrm{i}\sin(-n\theta)$ ***(M1)***

$= \cos n\theta - \mathrm{i}\sin(n\theta)$ ***(A1)***

Therefore $z^n + \frac{1}{z^n} = 2\cos n\theta$ ***(AG)***

[2 marks]

(b) $\left(z+\frac{1}{z}\right)^4 = z^4 + 4z^3\left(\frac{1}{z}\right) + 6(z^2)\left(\frac{1}{z^2}\right) + 4z\left(\frac{1}{z^3}\right) + \frac{1}{z^4}$ ***(M1)***

$= z^4 + \frac{1}{z^4} + 4\left(z^2 + \frac{1}{z^2}\right) + 6$ ***(M1)***

$(2\cos\theta)^4 = 2\cos 4\theta + 8\cos 2\theta + 6$ ***(A1)***

$\cos^4\theta = \frac{1}{16}(2\cos 4\theta + 8\cos 2\theta + 6)$ ***(A1)***

$= \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$ ***(AG)***

[4 marks]

(c) (i) $\int_0^a \cos^4\theta \,\mathrm{d}\theta = \frac{1}{8}\int_0^a (\cos 4\theta + 4\cos 2\theta + 3)\,\mathrm{d}\theta$ ***(M1)***

$= \frac{1}{8}\left[\frac{1}{4}\sin 4\theta + 2\sin 2\theta + 3\theta\right]_0^a$ ***(A1)***

$g(a) = \frac{1}{8}\left(\frac{1}{4}\sin 4a + 2\sin 2a + 3a\right)$ ***(A1)***

(ii) $1 = \frac{1}{8}\left(\frac{1}{4}\sin 4a + 2\sin 2a + 3a\right)$

$a = 2.96$ ***(A1)***

Since $\cos^4\theta \ge 0$ then $g(a)$ is an increasing function so there is only one root. ***(R1)***

[5 marks]

Total [11 marks]

2. (a) M $(3\mu-2, \mu, 9-2\mu)$ ***(A1)***

[1 mark]

(b) (i) $\frac{x-4}{3}=\frac{y}{1}=\frac{z+3}{-2}$ or $\boldsymbol{r}=\begin{pmatrix}4\\0\\-3\end{pmatrix}+\lambda\begin{pmatrix}3\\1\\-2\end{pmatrix}$ ***(M1)(A1)***

(ii) $\vec{\mathrm{PM}}=\begin{pmatrix}3\mu-2-4\\ \mu\\ 9-2\mu+3\end{pmatrix}$ ***(M1)***

$=\begin{pmatrix}3\mu-6\\ \mu\\ 12-2\mu\end{pmatrix}$ ***(A1)***

[4 marks]

(c) (i) $\begin{pmatrix}3\mu-6\\ \mu\\ 12-2\mu\end{pmatrix}\cdot\begin{pmatrix}3\\1\\-2\end{pmatrix}=0$ ***(M1)***

$9\mu-18+\mu-24+4\mu=0$

$\mu=3$ ***(A1)***

(ii) $\vec{\mathrm{PM}}=\begin{pmatrix}3\\3\\6\end{pmatrix}$ ***(A1)***

$|\vec{\mathrm{PM}}|=\sqrt{3^2+3^2+6^2}$ ***(M1)***

$=3\sqrt{6}$ (accept $\sqrt{54}$ or 7.35) ***(A1)***

[5 marks]

(d) $\boldsymbol{n}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\3&3&6\\3&1&-2\end{vmatrix}=-12\boldsymbol{i}+24\boldsymbol{j}-6\boldsymbol{k}$ ***(M1)(A1)***

$=-6(2\boldsymbol{i}-4\boldsymbol{j}+\boldsymbol{k})$

$\begin{pmatrix}2\\-4\\1\end{pmatrix}\bullet\boldsymbol{r}=\begin{pmatrix}2\\-4\\1\end{pmatrix}\bullet\begin{pmatrix}4\\0\\-3\end{pmatrix}$ ***(M1)***

$2x-4y+z=5$ ***(A1)***

[4 marks]

continued…

Question 2 continued

(e) **EITHER**

l_1 is on π_1 from part (d).
Testing l_1 on π_2 gives $(3\mu - 2) - 5(\mu) - (9 - 2\mu) = -11$. ***(M1)***
Therefore l_1 is also on π_2 and is therefore the line of intersection. ***(R1)***

OR

$$\begin{array}{r} 2x - 4y + z = 5 \\ \underline{x - 5y - z = -11} \\ 3x - 9y = -6 \\ x - 3y = -2 \end{array}$$ ***(M1)***

If $y = \lambda, x = -2 + 3\lambda, z = -2\lambda + 9$ ***(A1)***

or $\dfrac{x+2}{3} = \dfrac{y}{1} = \dfrac{z-9}{-2}$ which is l_1.

[2 marks]

Total [16 marks]

3. (a) Rotation through B followed by a rotation through A can be represented as

$$\boldsymbol{T}_1\boldsymbol{T}_2 = \begin{pmatrix} \cos(A+B) & -\sin(A+B) \\ \sin(A+B) & \cos(A+B) \end{pmatrix}$$ ***(M1)***

But $\boldsymbol{T}_1\boldsymbol{T}_2 = \begin{pmatrix} \cos A & -\sin A \\ \sin A & \cos A \end{pmatrix}\begin{pmatrix} \cos B & -\sin B \\ \sin B & \cos B \end{pmatrix}$ ***(A1)***

$$= \begin{pmatrix} \cos A\cos B - \sin A\sin B & -\cos A\sin B - \sin A\cos B \\ \sin A\cos B + \cos A\sin B & -\sin A\sin B + \cos A\cos B \end{pmatrix}$$ ***(A1)(A1)***

Therefore $\sin(A+B) = \sin A\cos B + \cos A\sin B$ ***(AG)***

$\cos(A+B) = \cos A\cos B - \sin A\sin B$ ***(AG)***

[4 marks]

(b) $\tan 2A = \dfrac{\sin 2A}{\cos 2A}$ ***(M1)***

putting $B = A$

$\tan 2A = \dfrac{2\sin A\cos A}{\cos^2 A - \sin^2 A}$ ***(A1)(A1)***

Dividing numerator and denominator by $\cos^2 A$

$\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$ ***(M1)(A1)***

[5 marks]

(c) The transformation $\begin{pmatrix} \frac{5}{13} & -\frac{12}{13} \\ -\frac{12}{13} & -\frac{5}{13} \end{pmatrix}$ is a reflection in $y = x\tan\theta$

with $\cos 2\theta = \dfrac{5}{13}$ and $\sin 2\theta = -\dfrac{12}{13}$. ***(M1)***

$\tan 2\theta = -\dfrac{12}{5}$ ***(A1)***

$\dfrac{2\tan\theta}{1-\tan^2\theta} = -\dfrac{12}{5}$ ***(M1)***

$0 = 6\tan^2\theta - 5\tan\theta - 6$

$0 = (3\tan\theta + 2)(2\tan\theta - 3)$

$\tan\theta = -\dfrac{2}{3}$ or $\tan\theta = \dfrac{3}{2}$ ***(A1)***

As $\sin 2\theta < 0$, $\dfrac{\pi}{2} < \theta < \pi$, therefore $\tan\theta < 0$. ***(R1)***

So the equation of the line of reflection is $y = -\dfrac{2}{3}x$. ***(A1)***

> **Notes:** Accept any reasonable justification for the exclusion of $y = \dfrac{3}{2}x$.
> Award ***(R0)(A1)*** if both equations are given.

[6 marks]

Total [15 marks]

4. (a) X = length of Ian's throw.

$X \sim N(60.33, 1.95^2)$

$\mathrm{P}(X > x) = 0.80 \Rightarrow z = -0.8416$ ***(A1)***

$-0.8416 = \dfrac{x - 60.33}{1.95}$ ***(M1)***

$x = 58.69$ m ***(A1)*** ***(N3)***

[3 marks]

(b) Y = length of Karl's throw.

$Y \sim N(59.39, \sigma^2)$

$\mathrm{P}(Y > 56.52) = 0.80 \Rightarrow z = -0.8416$ ***(A1)***

$-0.8416 = \dfrac{56.52 - 59.39}{\sigma}$ ***(M1)***

$\sigma = 3.41$ (accept 3.42) ***(A1)***

[3 marks]

(c) (i) $Y \sim N(59.50, 3.00^2)$

$X \sim N(60.33, 1.95^2)$

EITHER

$\mathrm{P}(Y \geq 65) = 0.0334$ $\quad \mathrm{P}(X \geq 65) = 0.00831$ (no **(AP)** here) ***(A2)(A2)***

OR

$\mathrm{P}(Y \geq 65) = \mathrm{P}\left(Z \geq \dfrac{65 - 59.50}{3.00}\right)$ ***(M1)***

$= \mathrm{P}(Z \geq 1.833)$

$= 0.0334$ (accept 0.0336) ***(A1)***

$\mathrm{P}(X \geq 65) = \mathrm{P}\left(Z \geq \dfrac{65 - 60.33}{1.95}\right)$ ***(M1)***

$= \mathrm{P}(Z \geq 2.395)$

$= 0.0083$ (accept 0.0084) ***(A1)***

THEN

Karl is more likely to qualify since $\mathrm{P}(Y \geq 65) > \mathrm{P}(X \geq 65)$. ***(R1)***

Note:	Award full marks if probabilities are not calculated but the correct conclusion is reached with the reason 1.833 < 2.395.

continued…

Question 4 (c) continued

(ii) If p represents the probability that an athlete throws 65 metres or more then with 3 throws the probability of qualifying for the final is $1-(1-p)^3$, or $p+(1-p)p+(1-p)^2 p$,
or $p^3+3(1-p)p^2+3(1-p)^2 p$. ***(M1)***

Therefore $\mathrm{P}(\text{Ian qualifies}) = 1-(1-0.00831)^3$

$= 0.0247$ ***(A1)***

$\mathrm{P}(\text{Karl qualifies}) = 1-(1-0.0334)^3$

$= 0.0969$ ***(A1)***

Assuming independence ***(R1)***

$\mathrm{P}(\text{both qualify}) = (0.0247)(0.0969)$ ***(M1)***

$= 0.00239$ ***(A1)***

Note:	Depending on use of tables or gdc answers may vary from 0.00239 to 0.00244.

[11 marks]

Total [17 marks]

5. (a) $(e^{2\theta}+1)\,dy = y\,d\theta$

Separating variables yields $\int \frac{dy}{y} = \int \frac{d\theta}{e^{2\theta}+1}$ ***(M1)***

$x = e^{\theta} \Rightarrow e^{2\theta}+1 = x^2+1$ ***(A1)***

$\frac{dx}{d\theta} = e^{\theta}$ ***(A1)***

$d\theta = \frac{dx}{x}$

$\int \frac{dy}{y} = \int \frac{dx}{x(x^2+1)}$ ***(AG)***

[3 marks]

(b) Using partial fractions let

$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$ ***(M1)***

$A(x^2+1) + Bx^2 + Cx = 1$

$A = 1, B = -1, C = 0$ ***(A1)***

$\int \frac{1}{x(x^2+1)}dx = \int \left(\frac{1}{x} - \frac{x}{x^2+1}\right)dx$

$= \ln x - \frac{1}{2}\ln(x^2+1) + C$ ***(A1)(A1)***

[4 marks]

(c) Therefore $\ln y = \ln x - \frac{1}{2}\ln(x^2+1) + \ln k$

$\ln y = \ln\left(\frac{kx}{\sqrt{x^2+1}}\right)$ ***(A1)***

$y = \frac{kx}{\sqrt{x^2+1}}$

When $\theta = 0, x = 1, y = \sqrt{2} \Rightarrow \sqrt{2} = \frac{k}{\sqrt{2}} \Rightarrow k = 2$. ***(M1)***

Therefore $y = \frac{2e^{\theta}}{\sqrt{e^{2\theta}+1}}$. ***(A2)***

[4 marks]

Total [11 marks]

Note: In this question do not penalize answers given to more than three significant figures.

6. (i) (a) Let $\lambda = \mathrm{E}(X) = \mathrm{Var}(X)$. Then $\lambda = \lambda^2 - 6$. ***(M1)***

Therefore $\lambda = \dfrac{1 \pm \sqrt{25}}{2}$ ***(M1)***

and since λ must be positive $\lambda = 3$. ***(A1)***

[3 marks]

(b) Then $\mathrm{P}(X \le 3) = 0.647$. ***(A1)*** ***(N1)***

[1 mark]

(c) $\mathrm{E}(X+Y) = 3+2 = 5$. Since X and Y are independent $X + Y$ has a Poisson distribution with mean = 5. ***(M1)***

Hence $\mathrm{P}(X+Y<4) = 0.265$. ***(A1)*** ***(N1)***

Note: Award ***(N0)*** if $\mathrm{P}(X+Y \le 4)$ is given instead.

[2 marks]

(d) (i) $\mathrm{E}(U) = \mathrm{E}(X) + 2\mathrm{E}(Y) = 7$ ***(A1)***

$\mathrm{Var}(U) = \mathrm{Var}(X) + 4\,\mathrm{Var}(Y) = 11$ ***(A1)***

(ii) U does not have a Poisson distribution ***(A1)***

because $\mathrm{Var}(U) \ne \mathrm{E}(U)$. ***(R1)***

[4 marks]

(ii) (a) This is a two-tailed interval so we need $z_{0.975} = 1.96$. ***(M1)(A1)***

We must have $\dfrac{1.96\sigma}{\sqrt{n}} = 20$. ***(M1)***

Therefore $1.96 \times 100 = 20\sqrt{n}$. ***(A1)***

This yields $\sqrt{n} \ge 9.8 \Rightarrow n \ge 9.8^2 = 96$ (Accept 97). ***(A1)***

[5 marks]

(b) In this case $z_\alpha = \dfrac{\sqrt{166}}{5} = 2.577$ so $\alpha = 0.995$ ***(A1)***

and since this is a two-tailed test the level of confidence is given by

$0.995 - 0.005 = 0.99\cdot$ ***(M1)***

i.e. 99.0% (Accept 99%) ***(A1)***

[3 marks]

continued…

Question 6 continued

(iii) (a) $p_1 = 0.0784$
$p_2 = 0.2160$
$p_3 = 0.2960$
$p_4 = 0.280$
$p_5 = 0.1296$ ***(A4)***

Note: Award ***[4 marks]*** if all answers are correct, ***[3 marks]*** if one answer is wrong, ***[2 marks]*** if two answers are wrong and ***[0 marks]*** if three or more answers are wrong.

[4 marks]

(b) This is a test of the hypothesis :
H_0 : The assumption of the physicist is correct.
H_1 : The assumption of the physicist is not correct. ***(R1)***
It calls for a χ^2 test. ***(M1)***

From (a) we can compute the expected values $e_j = p_j \times 40$ ***(M1)***
$e_1 = 3.136$
$e_2 = 8.64$
$e_3 = 11.84$
$e_4 = 11.2$
$e_5 = 5.184$ ***(A1)*** ***(N1)***

Note: Award ***[0 marks]*** if at least one answer is incorrect.

The first two intervals must be combined. ***(M1)***

Note: If the first two intervals are not combined award ***(M0)*** and **ft** marks according to the markscheme, $\chi^2_{4,\,0.95} = 9.488$.

$$\chi^2_{\text{calc}} = \frac{(14-11.776)^2}{11.776} + \frac{(9-11.84)^2}{11.84} + \frac{(8-11.2)^2}{11.2} + \frac{(9-5.184)^2}{5.184} = 4.8245$$ ***(A1)***

$\chi^2_{3,\,0.95} = 7.815$ ***(A1)***

The hypothesis H_0 can be accepted because $\chi^2_{\text{calc}} < \chi^2_{3,\,0.95}$. ***(R1)***

[8 marks]

Total [30 marks]

7. (i) (a) R is reflexive because $|z| = |z| \Rightarrow zRz$. *(A1)*

R is symmetric because $(|z_1| = |z_2| \Rightarrow |z_2| = |z_1|) \Rightarrow (z_1Rz_2 \Rightarrow z_2Rz_1)$ *(A1)*

R is transitive because $(|z_1| = |z_2| \text{ and } |z_2| = |z_3| \Rightarrow |z_1| = |z_3|)$

$\Rightarrow (z_1Rz_2 \text{ and } z_2Rz_3 \Rightarrow z_1Rz_3)$ *(A1)*

[3 marks]

(b) In the Argand diagram this corresponds to the concentric circles *(A1)*
centered at the origin. *(A1)*

[2 marks]

(ii) (a) The operation table is thus:

*	1	3	4	9	10	12
1	1	3	4	9	10	12
3	3	9	12	1	4	10
4	4	12	3	10	1	9
9	9	1	10	3	12	4
10	10	4	1	12	9	3
12	12	10	9	4	3	1

(A4)

Note: Award ***(A3)*** if one entry is incorrect, ***(A2)*** if two entries are incorrect, ***(A1)*** if three are incorrect, ***(A0)*** if four or more are incorrect.

[4 marks]

(b) $*$ is associative and commutative since multiplication modulo 13 is associative and commutative *(A1)*
The set is closed under $*$ *(A1)*
1 is the identity element *(A1)*
Every element has an inverse because 1 is on each row (or on each column). *(A1)*

[4 marks]

(c) 1 is of order 1
12 is of order 2 *(A1)*
3 and 9 are of order 3 *(A1)*
4 and 10 are of order 6 *(A1)*

Note: If one answer is wrong, award ***(A1)***, if two or more answers are wrong award ***(A0)***.

[3 marks]

(d) There are four subgroups:
{1}
{1, 12} *(A1)*
{1, 3, 9} *(A2)*
{1, 3, 4, 9, 10, 12}

[3 marks]

continued…

Question 7 continued

(iii) Let $a^{-1} = b$ ***(M1)***
Then $e = b \times a = b \times a \times a$ ***(M1)***
so that $e = (b \times a) \times a = e \times a$ ***(M1)***
and therefore $e = a$ ***(M1)(AG)***

Note: There are other correct solutions.

[4 marks]

(iv) (a) A cyclic group is a group which is generated by one of its elements (or words to that effect). ***(M2)***

[2 marks]

(b) We can assume that (G, #) has at least two elements and hence contains an element, say b, which is different from e, its identity. ***(R1)***
The order of b is equal to the order q of the subgroup it generates. ***(M1)***
By Lagrange's theorem q must be a factor of p and since p is prime either $q = 1$ or $q = p$. ***(R1)***
Since $b \neq e$ we see that $q \neq 1$ and therefore $q = p$. ***(R1)***
But if the order of b is p then b generates (G, #) which is therefore cyclic. ***(R1)***

[5 marks]

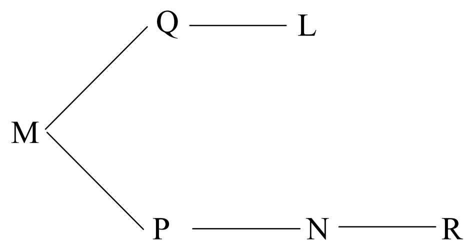
Total [30 marks]

8. (i) (a) $M \to Q$ ***(M1)***

$Q \to L$ ***(A1)***

$M \to P$ ***(A1)***

$P \to N \to R$ ***(A1)***



(A1)

Note: There are other correct answers.

[5 marks]

(b) The total weight is $2 + 1 + 3 + 2 + 3 = 11$. ***(A1)***

[1 mark]

(ii) (a) G_2 does not have an Eulerian trail because four vertices have an odd order. ***(A1)(R1)***

Note: There are other correct answers (*e.g.* only two vertices have an even order).

[2 marks]

(b) The Eulerian trail must start and end at a vertex with odd order, here *B* and *C* (or *C* and *B*). ***(R1)***

$B \to A \to B \to C \to E \to C \to F \to E \to F \to B \to D \to E \to D$
$\to A \to D \to C$ ***(A3)***

Notes: Award ***(A2)*** if there is one mistake, ***(A1)*** if there are two mistakes and ***(A0)*** if there are more than two mistakes. There are many other correct answers.

[4 marks]

(iii) (a) $r^2 - 2r + 2 = 0$ ***(A1)***

$r = \dfrac{2 \pm \mathrm{i}\sqrt{4}}{2}$ ***(A1)***

$= 1 \pm \mathrm{i}$ ***(AG)***

[2 marks]

(b) $x_n = C_1(1+\mathrm{i})^n + C_2(1-\mathrm{i})^n$ ***(A1)***

[1 mark]

(c) $C_1(1+\mathrm{i}) + C_2(1-\mathrm{i}) = 1$ ***(A1)***

$C_1(1+\mathrm{i})^2 + C_2(1-\mathrm{i})^2$

$= 2\mathrm{i}C_1 - 2\mathrm{i}C_2 = 2$ ***(A1)***

$C_1 = -\dfrac{\mathrm{i}}{2} \qquad C_2 = \dfrac{\mathrm{i}}{2}$ ***(M1)(A1)***

$x_n = \dfrac{\mathrm{i}}{2}\left[-(1+\mathrm{i})^n + (1-\mathrm{i})^n\right]$ ***(A1)***

$= \mathrm{i}\dfrac{\left(\sqrt{2}\right)^n}{2}\left[\mathrm{e}^{-\frac{\mathrm{i}n\pi}{4}} - \mathrm{e}^{\frac{\mathrm{i}n\pi}{4}}\right]$ ***(M1)(A1)***

$= \dfrac{\mathrm{i}}{2}\left(\sqrt{2}\right)^n\left(-2\mathrm{i}\sin\dfrac{n\pi}{4}\right)$ ***(A1)***

$= \left(\sqrt{2}\right)^n \sin\dfrac{n\pi}{4}$ ***(AG)***

Note: There are other equivalent correct approaches.

[8 marks]

(iv) (if) $r = kpq + p + q \Rightarrow r = (kp+1)q + p \Leftrightarrow r \equiv p (\mathrm{mod}\, q)$ ***(A1)***

$\Rightarrow r = (hq+1)p + q \Leftrightarrow r \equiv q (\mathrm{mod}\, p)$ ***(A1)***

(only if) $r = mp + q = nq + p$ ***(A1)***

Hence $(m-1)p = (n-1)q$ and since p and q are relatively prime, p divides $n - 1$. ***(M1)(A1)***

Therefore $n = sp + 1$. ***(A1)***

Consequently $r = (sp+1)q + p = spq + p + q \Leftrightarrow r \equiv p + q (\mathrm{mod}\, pq)$. ***(A1)***

[7 marks]

Total [30 marks]

9. (i) The ratio test must be used. ***(M1)***

The series must converge if for n sufficiently large

$$\frac{|x|^{n+1}(n+1)}{|x|^{n}(n+1+1)} = \frac{|x|(n+1)}{n+1+1} < 1$$

that is to say if for n sufficiently large

$$|x| < \frac{n+1}{n+1+1}$$ ***(M1)***

This will be the case if $|x| < 1$. ***(A1)***

The series is divergent when $x = 1$ because it is equivalent to the harmonic series. ***(M1)(A1)***

The series is convergent when $x = -1$ because it is alternating with a general term whose absolute value **decreases** to 0. ***(M1)***

Therefore range $[-1, 1[$ ***(A1)***

[7 marks]

continued…

Question 9 continued

(ii) (a) $f'(x) = 1 - x^2$ so that it is negative in the interval and hence $f(x)$ is decreasing in the interval. ***(M1)***

Hence the maximum of *f* in the interval $\left[1, \frac{4}{3}\right]$ is $f(1) = \frac{4}{3}$. ***(M1)(A1)***

The minimum of *f* in the interval $\left[1, \frac{4}{3}\right]$ is $f\left(\frac{4}{3}\right) = \frac{98}{81} > 1$. ***(M1)(A1)***

[5 marks]

(b) $f(x) = x \Leftrightarrow 2 - x^3 = 0 \Leftrightarrow a = 2^{\frac{1}{3}}$ ***(A1)***

[1 mark]

(c) It follows from (a) that for all n, $1 \le x_n \le \frac{4}{3}$. ***(M1)***

f is continuous and differentiable on the interval $\left[1, \frac{4}{3}\right]$. ***(M1)***

Hence from the mean value theorem it follows that

$x_{n+1} - a = f(x_n) - f(a) = (x_n - a)f'(\xi)$ with $1 \le \xi \le \frac{4}{3}$. ***(M1)***

In $\left[1, \frac{4}{3}\right]$, $f'(\xi) = 1 - \xi^2$ is bounded in absolute value by $\frac{7}{9}$. ***(A1)***

Therefore $|x_{n+1} - a| \le |x_n - a||f'(\xi)| \le |x_n - a| \times \frac{7}{9}$. ***(A1)***

[5 marks]

(d) $|x_0 - a| = |1 - a| < 1$ ***(A1)***

Hence it follows from (c) that for all positive integers n, $|x_n - a| \le \left(\frac{7}{9}\right)^n$. ***(A1)***

Since $\left(\frac{7}{9}\right)^n$ goes to 0 when n goes to infinity ***(A1)***

it follows that $|x_n - a|$ goes to 0 as n goes to infinity which means that x_n converges to *a*. ***(A1)***

[4 marks]

continued…

Question 9 continued

(iii) (a) For all n, $\sin\frac{2\pi}{n^2} < \frac{2\pi}{n^2}$ so using the comparison test ***(M1)***

since the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent ***(M1)***

then the series under consideration is also convergent. ***(A1)***

[3 marks]

(b) $\frac{1}{n^2} < \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$ ***(M1)(AG)***

[1 mark]

(c) All the terms of the series are non negative. ***(A1)***
The first term is $= 0$ and the second is $= 1$. Hence the sum of the series is ≥ 1. ***(A1)***
Since every term of the series from $n = 2$ on are bounded by $2\pi\left(\frac{1}{n-1} - \frac{1}{n}\right)$, the sum of the series will be bounded by the sum of these terms, *i.e.* by $2\pi \times \left(1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \ldots\right) = 2\pi$.

So $0 \leq \sin\frac{2\pi}{n^2} \leq \frac{2\pi}{n^2} < 2\pi\left(\frac{1}{n-1} - \frac{1}{n}\right)$ ***(M1)(A1)***

So $1 \leq S < 2\pi$ ***(AG)***

[4 marks]

Total [30 marks]

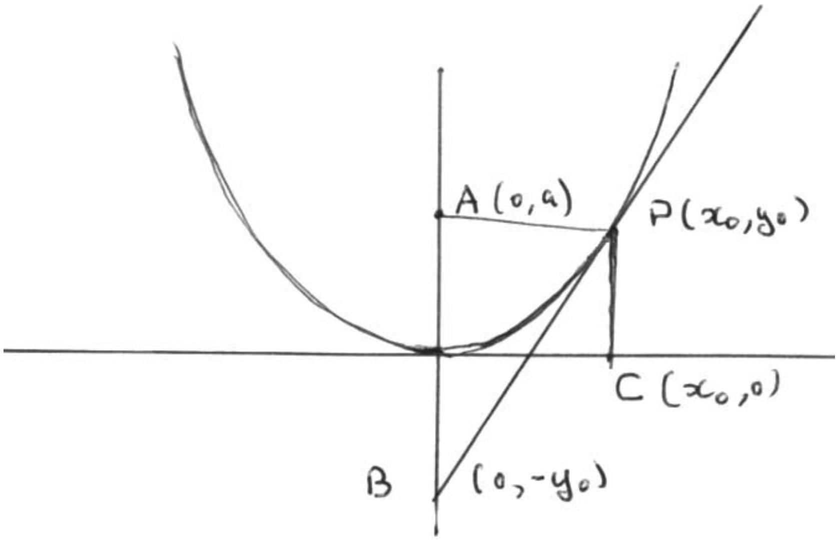
10. (i) (a) The equality of the distances gives: $(y-a)^2 + x^2 = (y-b)^2$. ***(M1)(A1)***

Hence $-2ay + 4py + a^2 = -2by + b^2$ and since this must be true for all values of y it follows that: $-2a + 4p = -2b$ and $a^2 - b^2 = 0$. ***(M1)(A1)***

Since $p > 0$ the second condition leads to $a + b = 0$ so that finally $a = p$ and $b = -p$. ***(M1)(A1)***

[6 marks]

(b)



The equation of the tangent to the parabola at the point $\mathrm{P}(x_0, y_0)$ is

$y - y_0 = \dfrac{x_0(x - x_0)}{2p}$. ***(M1)(A1)***

Therefore the tangent intersects the y-axis at the point $\mathrm{B}(0, -y_0)$. ***(A1)***

If C is the point $(x_0, 0)$ then $\mathrm{A\hat{B}P} = \mathrm{B\hat{P}C}$. ***(A1)***

$\mathrm{AB}^2 = (y_0 + p)^2 = (y_0 - p)^2 + 4py_0 = \mathrm{AP}^2$ and hence the triangle $\Delta\mathrm{APB}$ is isosceles so that $\mathrm{A\hat{P}B} = \mathrm{A\hat{B}P} = \mathrm{B\hat{P}C}$ which establishes the result. ***(M1)(A1)***

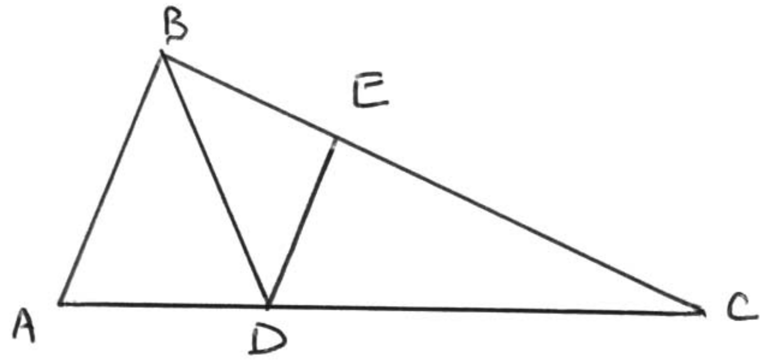
Note:	There are many other solutions.

[6 marks]

continued…

Question 10 continued

(ii)



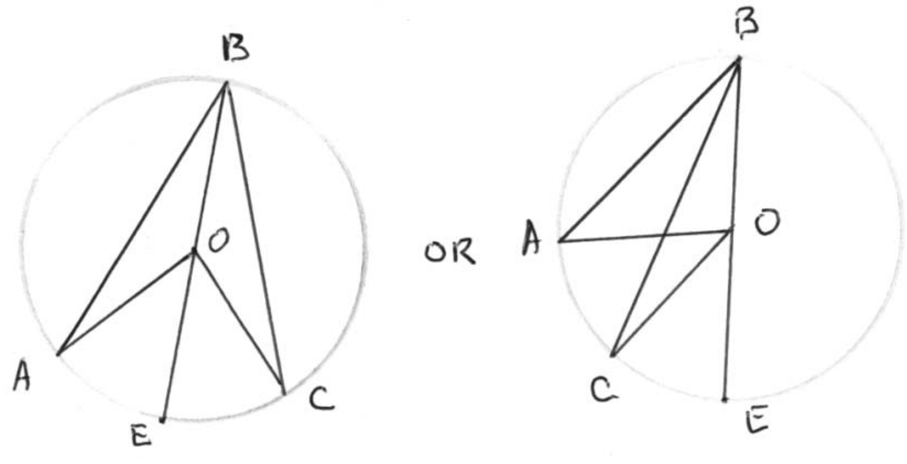
Let E be the intersection of (BC) and of a line parallel to (AB) passing through D. Then $B\hat{D}E = D\hat{B}A = D\hat{B}E$ so that ΔBDE is isosceles and hence BE = DE. ***(M1)(A1)***

Then $\dfrac{AD}{DC} = \dfrac{BE}{EC} = \dfrac{DE}{EC} = \dfrac{AB}{BC}$ ***(M1)(A1)***

Note: There are other solutions.

[4 marks]

(iii) (a)



Let E be the point of the circle such that [EB] is a diameter.

Then ΔBCO and ΔABO are isosceles so that $E\hat{B}C = O\hat{B}C = B\hat{C}O$ and $A\hat{B}E = A\hat{B}O = O\hat{A}B$. ***(A1)***

Hence $E\hat{O}A = O\hat{B}A + O\hat{A}B = 2A\hat{B}E$

and $E\hat{O}C = B\hat{C}O + O\hat{B}C = 2E\hat{B}C$. ***(A1)***

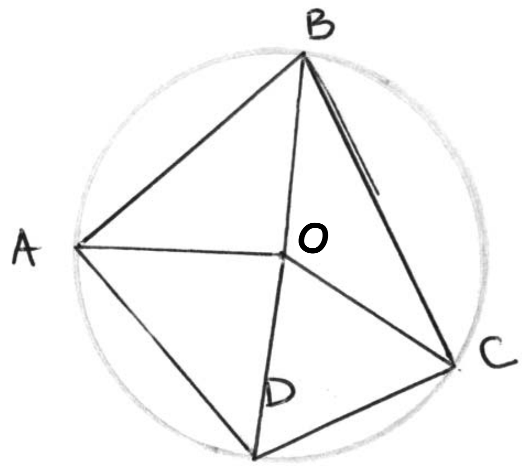
Adding (or substracting, according to whether O is inside or outside ΔABC) we see that $A\hat{O}C = 2A\hat{B}C$. ***(M1)(A1)***

[4 marks]

continued…

Question 10 (iii) continued

(b)



From (a) $2\text{A}\hat{\text{B}}\text{C}\ (= 2\text{A}\hat{\text{B}}\text{D} + 2\text{D}\hat{\text{B}}\text{C}) = \text{A}\hat{\text{O}}\text{D} + \text{D}\hat{\text{O}}\text{C}$

and $2\text{C}\hat{\text{D}}\text{A}\ (= 2\text{C}\hat{\text{D}}\text{B} + 2\text{B}\hat{\text{D}}\text{A}) = \text{C}\hat{\text{O}}\text{B} + \text{B}\hat{\text{O}}\text{A}$. ***(M1)(A1)***

Note:	The expressions inside of the parenthesis are only there to identify which of the two possible angles $\text{A}\hat{\text{O}}\text{C}$ are considered and need not be included in the candidate script.

Finally $2\text{A}\hat{\text{B}}\text{C} + 2\text{C}\hat{\text{D}}\text{A} = \text{A}\hat{\text{O}}\text{D} + \text{D}\hat{\text{O}}\text{C} + \text{C}\hat{\text{O}}\text{B} + \text{B}\hat{\text{O}}\text{A} = 360^\circ$ ***(A1)***

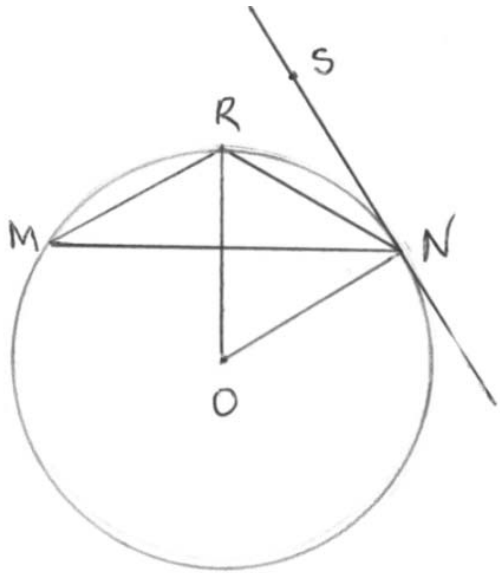
so that $\text{A}\hat{\text{B}}\text{C} + \text{C}\hat{\text{D}}\text{A} = 180^\circ$. ***(AG)***

[3 marks]

continued…

Question 10 continued

(iv)



Let S be any point on the tangent, different from N, as shown in the figure above. Then

$\mathrm{R\hat{O}N} = 2\mathrm{R\hat{M}N} = 2\mathrm{R\hat{N}M}$ ***(M1)(A1)***

But $\mathrm{R\hat{O}N} = 180^\circ - 2\mathrm{R\hat{N}O}$ since $\Delta\mathrm{RNO}$ is isosceles ***(M1)***

so that $\mathrm{R\hat{N}O} = 90^\circ - \dfrac{\mathrm{R\hat{O}N}}{2} = 90^\circ - \mathrm{R\hat{N}M}$. ***(A1)***

Also $\mathrm{R\hat{N}S} = 90^\circ - \mathrm{R\hat{N}O}$. ***(A1)***

Therefore $\mathrm{R\hat{N}M} = 90^\circ - \mathrm{R\hat{N}O} = \mathrm{R\hat{N}S}$. ***(M1)(A1)***

Note: There are other solutions.

[7 marks]

Total [30 marks]